

Approximation by Kantorovich type (p, q) -Bernstein-Schurer Operators

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Abstract

In this paper, we introduce a Shurer type generalization of (p, q) -Bernstein-Kantorovich operators based on (p, q) -integers and we call it as (p, q) -Bernstein-Schurer Kantorovich operators. We study approximation properties for these operators based on Korovkin's type approximation theorem and also study some direct theorems. Furthermore, we give comparisons and some illustrative graphics for the convergence of operators to some function.

Keywords and phrases: (p, q) -Bernstein-Schurer Kantorovich operators; (p, q) -Bernstein Kantorovich operators; q -Bernstein-Schurer Kantorovich operators; modulus of continuity; Positive linear operator; Korovkin's type approximation theorem.

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1 Introduction and preliminaries

The applications of q -calculus emerged as a new area in the field of approximation theory from last two decades. The development of q -calculus has led to the discovery of various modifications of Bernstein polynomials involving q -integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [16] introduced the first q -analogue of Bernstein operators [5] and investigated its approximating and shape-preserving properties. Another q -generalization of the classical Bernstein polynomials is due to Phillips [23]. Several generalizations of well-known positive linear operators based on q -integers were introduced and their approximation properties have been studied by several authors. For instance, the approximation properties of q -Bleimann, Butzer and Hahn operators [4]; q -analogue of Szász-Kantorovich operators [17]; Approximation by Kantorovich type q -Bernstein operators [8] and q -analogue of generalized Bernstein-Shurer operators [18] were studied.

Mursaleen et al studied approximation properties of the q -analogue of generalized Bernstein Shurer operators [18].

Recently, Mursaleen et al applied (p, q) -calculus in approximation theory and introduced (p, q) -analogue of Bernstein Operators [20] and (p, q) -analogue of Bernstein-Stancu Operators [21], (p, q) -analogue of Bernstein-Kantorovich Operators [22] respectively. Recently these works got attention in computer aided geometric design (CAGD)

As in computer aided geometric design , the basis of Bernstein polynomials plays a significant role in order to preserve the shape of the curves or surfaces. The classical Bezier curve [6] and q -bezier [24, 25] constructed with Bernstein basis functions are the most important curve in CAGD. Thus motivated by the work of Mursaleen et al in [20], very recently, Khalid et al in [19] introduced Bezier curves and surfaces in Computer aided geometric design defined by (p, q) -integers.

Ozarslan and Vedi [1] introduced Kantorovich type genelaization of the q -Bernstein-Schurer operators and studied some approximation properties of these operators. These operators are given as follows:

$$K_{n,\ell}^q(f; x) = \sum_{k=0}^{n+\ell} b_{n,\ell,k}^q(x) \int_0^1 f \left(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t \right) d_q t, \quad x \in [0, 1], \quad (1.1)$$

$$b_{n,\ell,k}^q(x) := \begin{bmatrix} n+\ell \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n+\ell-k-1} (1 - q^s x).$$

Where $\ell \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 < q < 1$ is fixed and $K_{n,\ell}^q : C[0, 1 + \ell] \rightarrow C[0, 1 + \ell]$ are defined for all $n \in \mathbb{N}$ and for any function $f \in C[0, 1 + \ell]$.

Details on the q -calculus can be found in [11] and for the applications of q -calculus in approximation theory, one can refer [2].

Mursaleen et al introduced (p, q) -Bernstein-kantorovich operators as follows in [22]

$$K_n^{(p,q)}(f; x) = [n+1]_{p,q} \sum_{k=0}^n \frac{(p-q)}{p^k(p-1) - q^k(k-1)} b_{n,k}^{(p,q)}(x) \int_{\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} f(t) d_{p,q} t, \quad x \in [0, 1]$$

where

$$b_{n,k}^{(p,q)}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k} = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x).$$

Motivated by the work of Mursaleen et al for (p, q) -Bernstein-Kantorovich operators based on (p, q) -integers in [22], now we present a Shurer type generalisation of (p, q) -Bernstein-Kantorovich operators.

Let us recall certain notations of (p, q) -calculus:

The (p, q) -integer was introduced in order to generalize or unify several forms of q -oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [7]. The (p, q) -integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \dots, \quad 0 < q < p \leq 1.$$

The (p, q) -Binomial expansion is

$$(ax + by)_{p,q}^n := \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\frac{k(k-1)}{2}} p^{\frac{(n-k)(n-k-1)}{2}} a^{n-k} b^k x^{n-k} y^k$$

$$(x + y)_{p,q}^n := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y).$$

Also, the (p, q) -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}$$

and the definite integrals of the function f are defined by

$$\int_0^a f(x) d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \quad \text{when } \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \text{when } \left|\frac{p}{q}\right| > 1.$$

Details on (p, q) -calculus can be found in [10, 14, 15, 26, 27]. For $p = 1$, all the notions of (p, q) -calculus are reduced to q -calculus.

Now, we introduce a new generalization of q -Bernstein-Kantorovich operators by using the notion of (p, q) -calculus and call it as (p, q) -Bernstein-Schurer-Kantorovich operators. We study the approximation properties based on Korovkin's type approximation theorem and also establish some direct theorems. Further, we show comparisons and some illustrative graphics for the convergence of operators to a function.

2 Construction of Operators

Now, we introduce (p, q) -analogue of Kantorovich type Bernstein-Schurer operators as

$$K_{n,\ell}^{(p,q)}(f; x) = \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 f\left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t\right) d_{p,q}t, \quad x \in [0, 1], \quad (2.1)$$

where

$$b_{n,\ell,k}^{(p,q)}(x) = \begin{bmatrix} n+\ell \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n+\ell-k} = \begin{bmatrix} n+\ell \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n+\ell-k-1} (p^s - q^s x).$$

For $p = 1$, equation (2.1) turns out to be the classical q -Bernstein-Kantorovich operators (1.1).

Now, we have the following basic lemmas:

Lemma 2.1 For $x \in [0, 1]$, $0 < q < p \leq 1$

$$(i) \quad K_{n,\ell}^{(p,q)}(1; x) = 1;$$

$$(ii) \quad K_{n,\ell}^{(p,q)}(t; x) = \frac{(px+1-x)_{p,q}^{n+\ell}}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+2q-1)[n+\ell]_{p,q}}{[2]_{p,q}[n+1]_{p,q}} x;$$

$$(iii) \quad K_{n,\ell}^{(p,q)}(t^2; x) = \frac{(p^2x+1-x)_{p,q}^{n+\ell}}{[3]_{p,q}[n+1]_{p,q}^2} + \left(1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}}\right) \frac{[n+\ell]_{p,q}}{[n+1]_{p,q}^2} (px+1-x)^{n+\ell-1} x \\ + \left(1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}}\right) \frac{[n+\ell]_{p,q}[n+\ell-1]_{p,q}}{[n+1]_{p,q}^2} x^2;$$

$$(iv) \quad K_{n,\ell}^{(p,q)}((t-x); x) = \frac{(p^2x+1-x)_{p,q}^{n+\ell}}{[2]_{p,q}[n+1]_{p,q}} + \left(\frac{(p+2q-1)}{[2]_{p,q}[n+1]_{p,q}} - 1\right) x;$$

$$(v) \quad K_{n,\ell}^{(p,q)}((t-x)^2; x) = \frac{(p^2x+1-x)_{p,q}^{n+\ell}}{[3]_{p,q}[n+1]_{p,q}^2} + \left\{ \left(1 + \frac{2q}{[2]_{p,q}} + \frac{q^2-1}{[3]_{p,q}}\right) \frac{[n+\ell]_{p,q}(px+1-x)^{n+\ell-1}}{[n+1]_{p,q}^2} - \frac{2(px+1-x)_{p,q}^{n+\ell}}{[2]_{p,q}[n+1]_{p,q}} \right\} x \\ + \left\{ q \left(1 + \frac{2(q-1)}{[2]_{p,q}} + \frac{(q-1)^2}{[3]_{p,q}}\right) \frac{[n+\ell]_{p,q}[n+\ell-1]_{p,q}}{[n+1]_{p,q}^2} - \frac{2(p+2q-1)[n+\ell]_{p,q}}{[2]_{p,q}[n+1]_{p,q}} + 1 \right\} x^2.$$

3 Approximation results

In this section, we find out the rate of convergence of the operators (2.1) using the modulus of continuity and Lipschitz classes. Furthermore, we calculate the rate convergence in terms of the first modulus of continuity of the function.

Let $C[0, 1 + \ell]$ be the linear space of all real valued continuous functions f on $[0, 1 + \ell]$ and let T be a linear operator defined on $C[0, 1 + \ell]$. We say that T is *positive* if for every non-negative $f \in C[0, 1 + \ell]$, we have $T(f, x) \geq 0$ for all $x \in [0, 1 + \ell]$.

The classical Korovkin approximation theorem [3, 13, ?] states as follows:

Let (T_n) be a sequence of positive linear operators from $C[0, 1 + \ell]$ into $C[a, b]$. Then $\lim_n \|T_n(f, x) - f(x)\|_{C[0, 1 + \ell]} = 0$, for all $f \in C[0, 1 + \ell]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_{C[0, 1 + \ell]} = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Theorem 3.1 Let $0 < q_n < p_n \leq 1$ such that $\lim_{n \rightarrow \infty} p_n = 1$ and $\lim_{n \rightarrow \infty} q_n = 1$. Then for each $f \in [0, 1 + \ell]$, $K_{n,\ell}^{(p_n, q_n)}(f; x)$ converges uniformly to f on $[0, 1 + \ell]$.

Proof. By the Korovkin Theorem it is sufficient to show that

$$\lim_{n \rightarrow \infty} \|K_{n,\ell}^{(p_n, q_n)}(t^m; x) - x^m\|_{C[0, 1 + \ell]} = 0, \quad m = 0, 1, 2.$$

By Lemma 2.1 (i), it is clear that

$$\lim_{n \rightarrow \infty} \|K_{n,\ell}^{(p_n, q_n)}(1; x) - 1\|_{C[0, 1 + \ell]} = 0.$$

Now, by Lemma 2.1 (ii)

$$|K_{n,\ell}^{(p_n, q_n)}(t; x) - x| \leq \frac{(p_n x + 1 - x)_{p_n, q_n}^{n+\ell}}{[2]_{p_n, q_n}[n+1]_{p_n, q_n}} + \left(\frac{(p_n + 2q_n - 1)}{[2]_{p_n, q_n}[n+1]_{p_n, q_n}} - 1\right) x$$

Taking maximum of both sides of the above inequality, we get

$$\|K_{n,\ell}^{(p_n,q_n)}(t; x) - x\|_{[0,1+\ell]} \leq \frac{p_n^n}{[2]_{p_n,q_n}[n+1]_{p_n,q_n}} + \frac{(p_n + 2q_n - 1)[n + \ell]_{p_n,q_n}}{[2]_{p_n,q_n}[n+1]_{p_n,q_n}} - 1$$

which yields

$$\lim_{n \rightarrow \infty} \|K_{n,\ell}^{(p_n,q_n)}(t; x) - x\|_{C[0,1+\ell]} = 0.$$

Similarly we can show that

$$\lim_{n \rightarrow \infty} \|K_{n,\ell}^{(p_n,q_n)}(t^2; x) - x^2\|_{C[0,1+\ell]} = 0.$$

Thus the proof is completed.

Now we will compute the rate of convergence in terms of modulus of continuity.

Let $f \in [0, 1 + \ell]$. The modulus of continuity of f denoted by $\omega_f(\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega_f(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|, \quad x, y \in [0, 1 + \ell].$$

It is known that $\lim_{\delta \rightarrow 0^+} \omega_f(\delta) = 0$ for $f \in C[0, 1 + \ell]$ and for any $\delta > 0$ one has

$$|f(y) - f(x)| \leq \omega_f(\delta) \left(\frac{|y - x|}{\delta} + 1 \right). \quad (3.1)$$

Theorem 3.2 If $f \in C[0, 1]$, then

$$|K_{n,\ell}^{(p,q)}(f; x) - f(x)| \leq 2\omega \left(f, \sqrt{\delta_{n,\ell}^{(p,q)}(x)} \right)$$

takes place, where $\delta_{n,\ell}^{(p,q)} = K_{n,\ell}^{(p,q)}((t - x)^2; x)$

Proof. Since $K_{n,\ell}^{(p,q)}(1; x) = 1$, we have

$$\begin{aligned} |K_{n,\ell}^{(p,q)}(f; x) - f(x)| &\leq K_{n,\ell}^{(p,q)}(|f(t) - f(x)|; x) \\ &= \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 f \left| \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t \right) - f(x) \right| d_{p,q}t. \end{aligned}$$

In view of (3.1), we get

$$\begin{aligned} |K_{n,\ell}^{(p,q)}(f; x) - f(x)| &\leq \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left(\frac{\left| \frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t - x \right|}{\delta} + 1 \right) \omega(f, \delta) d_{p,q}t \\ &= \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left| \frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t - x \right| d_{p,q}t \end{aligned}$$

Now using Cauchy-Schwartz inequality, we get

$$|K_{n,\ell}^{(p,q)}(f; x) - f(x)| \leq \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \times \left\{ \int_0^1 \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t - x \right)^2 d_{p,q} t \right\}^{\frac{1}{2}}.$$

Again applying the cauchy-Schwartz inequality, we have

$$\begin{aligned} |K_{n,\ell}^{(p,q)}(f; x) - f(x)| &\leq \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \left\{ \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \right. \\ &\quad \times \left. \int_0^1 \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t - x \right)^2 d_{p,q} t \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \right\}^{\frac{1}{2}} \\ &= \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \left\{ \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t - x \right)^2 d_{p,q} t \right\}^{\frac{1}{2}} \\ &= \omega(f, \delta) + \frac{\omega(f, \delta)}{\delta} \left\{ K_{n,\ell}^{(p,q)}((t-x)^2; x) \right\}^{\frac{1}{2}}. \end{aligned}$$

Now on taking $\delta := \delta_{n,\ell}^{(p,q)}(x) = K_{n,\ell}^{(p,q)}((t-x)^2; x)$, we obtain

$$|K_{n,\ell}^{(p,q)}(f; x) - f(x)| \leq 2\omega \left(f, \sqrt{K_{n,\ell}^{(p,q)}((t-x)^2; x)} \right)$$

This completes the proof of the theorem.

Now we give the rate of convergence of the operators $K_{n,\ell}^{(p,q)}$ in terms of the elements of the usual Lipschitz class $\text{Lip}_M(\alpha)$.

Let $f \in C[0, 1 + \ell]$, $M > 0$ and $0 < \alpha \leq 1$. We recall that f belongs to the class $\text{Lip}_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \leq M|t - x|^\alpha \quad (t, x \in [0, 1 + \ell])$$

is satisfied.

Theorem 3.3 Let $0 < q < p \leq 1$. Then for each $f \in \text{Lip}_M(\alpha)$ we have

$$|K_{n,\ell}^{(p,q)}(f; x) - f(x)| \leq M \left(\delta_{n,\ell}^{(p,q)}(x) \right)^{\frac{\alpha}{2}}$$

where

$$\delta_{n,\ell}^{(p,q)}(x) = K_{n,\ell}^{(p,q)}((t-x)^2; x).$$

Proof. By the monotonicity of the operators $K_{n,\ell}^{(p,q)}$, we can write

$$\begin{aligned}
|K_{n,\ell}^{(p,q)}(f; x) - f(x)| &\leq K_{n,\ell}^{(p,q)}(|f(t) - f(x)|; x) \\
&= \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 f \left| \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t \right) - f(x) \right| d_{p,q}t \\
&\leq M \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left| \left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t \right) - x \right|^\alpha d_{p,q}t
\end{aligned}$$

Now applying the Hölder's inequality for the sum with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ and taking into consideration Lemma 2.1(i) and Lemma 2.2(ii), we have

$$\begin{aligned}
|K_{n,\ell}^{(p,q)}(f; x) - f(x)| &\leq M \sum_{k=0}^{n+\ell} \left\{ b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left(\left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t \right) - x \right)^2 d_{p,q}t \right\}^{\frac{\alpha}{2}} \\
&\quad \times \left\{ b_{n,\ell,k}^{(p,q)}(x) \int_0^1 1 d_{p,q}t \right\}^{\frac{2-\alpha}{2}} \\
&\leq M \left\{ \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 \left(\left(\frac{[k]_{p,q}}{[n+1]_{p,q}} + \frac{[k+1]_{p,q} - [k]_{p,q}}{[n+1]_q} t \right) - x \right)^2 d_{p,q}t \right\}^{\frac{\alpha}{2}} \\
&\quad \times \left\{ \sum_{k=0}^{n+\ell} b_{n,\ell,k}^{(p,q)}(x) \int_0^1 1 d_{p,q}t \right\}^{\frac{2-\alpha}{2}} \\
&= M \{ K_{n,\ell}^{(p,q)}((t-x)^2; x) \}^{\frac{\alpha}{2}}.
\end{aligned}$$

Choosing $\delta_{n,\ell}^{(p,q)}(x) = K_{n,\ell}^{(p,q)}((t-x)^2; x)$, we arrive at the desired result.

Next we prove the local approximation property for the operators $K_{n,\ell}^{(p,q)}$. The Peetre's K -functional is defined by

$$K_2(f, \delta) = \inf \{ \|f - g\| + \delta \|g''\| : g \in W^2 \},$$

where

$$W^2 = \{g \in C[0, 1 + \ell] : g', g'' \in C[0, 1 + \ell]\}.$$

By [9], there exists a positive constant $C > 0$ such that $K_2(f, \delta) \leq C\omega_2(f, \delta^{\frac{1}{2}})$, $\delta > 0$; where the second order modulus of continuity is given by

$$\omega_2(f, \delta^{\frac{1}{2}}) = \sup_{0 < h \leq \delta^{\frac{1}{2}}} \sup_{x \in [0, 1 + \ell]} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Also for $f \in [0, 1 + \ell]$ the usual modulus of continuity is given by

$$\omega(f, \delta) = \sup_{0 < h \leq \delta^{\frac{1}{2}}} \sup_{x \in [0, 1 + \ell]} |f(x + h) - f(x)|.$$

Theorem 3.4 Let $f \in C[0, 1 + \ell]$ and $0 < q < p \leq 1$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that

$$|K_{n,\ell}^{(p,q)}(f; x) - f(x)| \leq C\omega_2(f, \sqrt{a_{n,\ell}^{(p,q)}(x)}) + \omega(f, c_{n,\ell}^{(p,q)}),$$

where

$$a_{n,\ell}^{(p,q)}(x) = K_{n,\ell}^{(p,q)}((t - x)^2; x) + \left(\alpha_{n,\ell}^{(p,q)}(x) - x\right)^2, \quad c_{n,\ell}^{(p,q)}(x) = \left(\alpha_{n,\ell}^{(p,q)}(x) - x\right)$$

and

$$\alpha_{n,\ell}^{(p,q)}(x) = \frac{(px + 1 - x)_{p,q}^{n+\ell}}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+2q-1)[n+\ell]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}x.$$

Proof. Using the operators (2.1), we define the following operators

$$\tilde{K}_{n,\ell}^{(p,q)}(f; x) := K_{n,\ell}^{(p,q)}(f; x) - f\left(\frac{(px + 1 - x)_{p,q}^{n+\ell}}{[2]_{p,q}[n+1]_{p,q}} + \frac{(p+2q-1)[n+\ell]_{p,q}}{[2]_{p,q}[n+1]_{p,q}}x\right) + f(x). \quad (3.2)$$

Then, by the Lemma 2.1 (ii) For $x \in [0, 1]$, $0 < q < p \leq 1$

$$\tilde{K}_{n,\ell}^{(p,q)}(1; x) = 1;$$

$$\tilde{K}_{n,\ell}^{(p,q)}(t - x; x) = 0.$$

Then, for a given $g \in W^2$. From Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u) g''(u) du, \quad t \in [0, A], \quad A > 0,$$

and for the operators (3.2), we get

$$\begin{aligned} \tilde{K}_{n,\ell}^{(p,q)}(g; x) &= g(x) + \tilde{K}_{n,\ell}^{(p,q)}\left(\int_x^t (t - u) g''(u) du; x\right), \\ | \tilde{K}_{n,\ell}^{(p,q)}(g; x) - g(x) | &= \left| \tilde{K}_{n,\ell}^{(p,q)}\left(\int_x^t (t - u) g''(u) du; x\right) \right| \\ &= \left| K_{n,\ell}^{(p,q)}\left(\int_x^t (t - u) g''(u) du; x\right) - \int_x^{\alpha_{n,\ell}^{(p,q)}(x)} \left(\alpha_{n,\ell}^{(p,q)}(x) - u\right) g''(u) du \right| \\ &\leq \left| K_{n,\ell}^{(p,q)}\left(\int_x^t (t - u) g''(u) du; x\right) \right| + \left| \int_x^{\alpha_{n,\ell}^{(p,q)}(x)} \left(\alpha_{n,\ell}^{(p,q)}(x) - u\right) g''(u) du \right| \\ &\leq \|g''\| K_{n,\ell}^{(p,q)}((t - x)^2; x) + \|g''\| \left(\alpha_{n,\ell}^{(p,q)}(x) - x\right)^2 \\ &= \|g''\| \left(K_{n,\ell}^{(p,q)}((t - x)^2; x) + \left(\alpha_{n,\ell}^{(p,q)}(x) - x\right)^2 \right) = \|g''\| a_{n,\ell}^{(p,q)}(x). \end{aligned}$$

On the other hand, by the definition of $\tilde{K}_{n,\ell}^{(p,q)}(f; x)$, we have

$$| \tilde{K}_{n,\ell}^{(p,q)}(f; x) | \leq 4\|f\|.$$

Now

$$\begin{aligned} | K_{n,\ell}^{(p,q)}(f; x) - f(x) | &\leq | \tilde{K}_{n,\ell}^{(p,q)}((f - g); x) - (f - g)(x) | + | \tilde{K}_{n,\ell}^{(p,q)}(g; x) - g(x) | \\ &\quad + \left| f(\alpha_{n,\ell}^{(p,q)}(x)) - f(x) \right| \\ &\leq 4\|f - g\| + a_{n,\ell}^{(p,q)}(x)\|g''\| + \left| f(\alpha_{n,\ell}^{(p,q)}(x)) - f(x) \right| \\ &\leq 4(\|f - g\| + a_{n,\ell}^{(p,q)}(x)\|g''\|) + \omega(f, c_{n,\ell}^{(p,q)}). \end{aligned}$$

In view of the property of K -functional, it can be easily seen that

$$\begin{aligned} | K_{n,\ell}^{(p,q)}(f; x) - f(x) | &\leq 4K(f, \sqrt{a_{n,\ell}^{(p,q)}(x)}) + \omega(f, c_{n,\ell}^{(p,q)}) \\ &\leq C\omega_2(f, \sqrt{a_{n,\ell}^{(p,q)}(x)}) + \omega(f, c_{n,\ell}^{(p,q)}). \end{aligned}$$

This completes the proof of the theorem.

4 Graphical analysis

With the help of Matlab, we show comparisons and some illustrative graphics [18] for the convergence of (p, q) -Bernstein-Schurer-Kantorovich operators $K_{n,\ell}^{(p,q)}$ to the function $f(x) = 1 + \cos(5x^2)$, for different values of parameters p, q, n , convergence of the operators $K_{n,\ell}^{(p,q)}$ to the function is shown in figure 1, 2, 3, and 4.

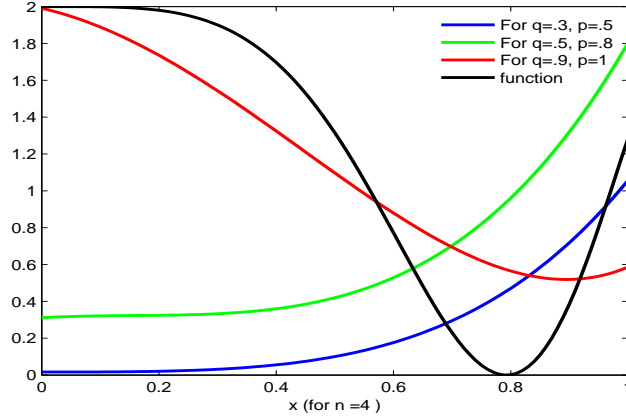


Figure 1: Approximation by (p, q) -Bernstein-Schurer-Kantorovich operators to a function

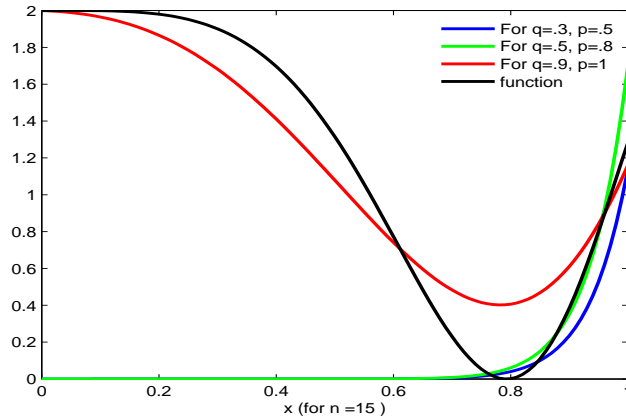


Figure 2: Approximation by (p, q) -Bernstein-Schurer-Kantorovich operators to a function

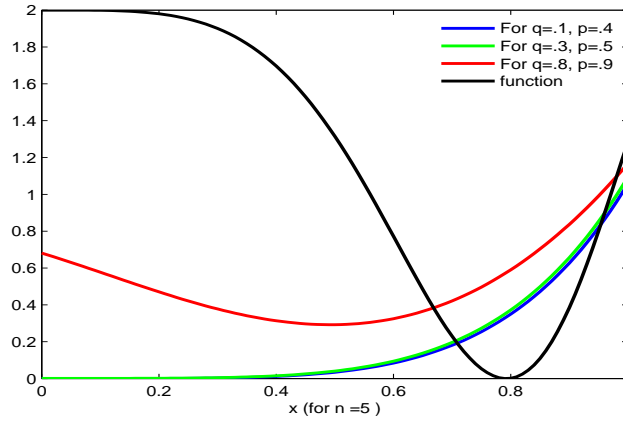


Figure 3: Approximation by (p, q) -Bernstein-Schurer-Kantorovich operators to a function

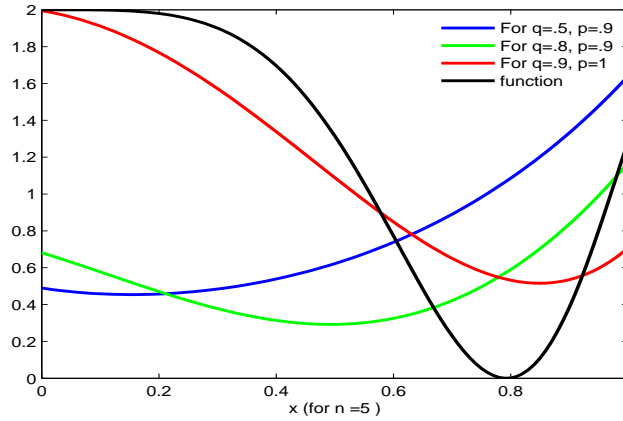


Figure 4: Approximation by (p, q) -Bernstein-Schurer-Kantorovich operators to a function

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